HIGHER ORDER ASYMPTOTIC FORMULAS FOR TOEPLITZ MATRICES WITH SYMBOLS IN GENERALIZED HÖLDER SPACES

ALEXEI YU. KARLOVICH

Abstract. We prove higher order asymptotic formulas for determinants and traces of finite block Toeplitz matrices generated by matrix functions belonging to generalized Hölder spaces with characteristic functions from the Bari-Stechkin class. We follow the approach of Böttcher and Silbermann and generalize their results for symbols in standard Hölder spaces.

1. Introduction

1.1. **Finite block Toeplitz matrices.** Let $\mathbb{Z}, \mathbb{N}, \mathbb{Z}_+$, and $\mathbb{C}$ be the sets of integers, positive integers, nonnegative integers, and all complex numbers, respectively. Suppose $N \in \mathbb{N}$. For a Banach space X, let X_N and $X_{N\times N}$ be the spaces of vectors and matrices with entries in X. Let $\mathbb{T}$ be the unit circle. For $1 \le p \le \infty$, let $L^p := L^p(\mathbb{T})$ and $H^p := H^p(\mathbb{T})$ be the standard Lebesgue and Hardy spaces of the unit circle. For $a \in L^1_{N\times N}$ one can define

$$a_k = \frac{1}{2\pi}\int_0^{2\pi} a(e^{i\theta})e^{-ik\theta}d\theta \quad (k \in \mathbb{Z}),$$

the sequence of the Fourier coefficients of a. Let I be the identity operator, P be the Riesz projection of L^2 onto H^2, $Q := I - P$, and define I, P, and Q on L^2_N elementwise. For $a \in L^\infty_{N\times N}$ and $t \in \mathbb{T}$, put $\widetilde{a}(t) := a(1/t)$ and $(Ja)(t) := t^{-1}\widetilde{a}(t)$. Define *Toeplitz operators*

$$T(a) := PaP|\operatorname{Im} P, \quad T(\widetilde{a}) := JQaQJ|\operatorname{Im} P$$

and *Hankel operators*

$$H(a) := PaQJ|\operatorname{Im} P, \quad H(\widetilde{a}) := JQaP|\operatorname{Im} P.$$

The function a is called the *symbol* of $T(a)$, $T(\widetilde{a})$, $H(a)$, $H(\widetilde{a})$. We are interested in the asymptotic behavior of *finite block Toeplitz matrices*

$$T_n(a) := (a_{j-k})_{j,k=0}^n$$

generated by (the Fourier coefficients of) the symbol a as $n \to \infty$. Many results about asymptotic properties of $T_n(a)$ as $n \to \infty$ are contained in the books by

2000 *Mathematics Subject Classification.* Primary 47B35; Secondary 15A15, 47B10, 47L20, 47A68.

Key words and phrases. Block Toeplitz matrix, determinant, trace, Szegő-Widom limit theorems, decomposing algebra, canonical Wiener-Hopf factorization, generalized Hölder space, Bari-Stechkin class.

The author is supported by F.C.T. (Portugal) grants SFRH/BPD/11619/2002 and FCT/FEDER/POCTI/MAT/59972/2004.

Grenander and Szegő [13], Böttcher and Silbermann [5, 6, 7], Hagen, Roch, and Silbermann [15], Simon [24], and Böttcher and Grudsky [2].

1.2. **Szegő-Widom limit theorems.** Let us formulate precisely the most relevant results. Let $(K_{2,2}^{1/2,1/2})_{N\times N}$ be the Krein algebra [19] of matrix functions a in $L^\infty_{N\times N}$ satisfying $\sum_{k=-\infty}^{\infty}\|a_k\|^2|k|<\infty$, where $\|\cdot\|$ is any matrix norm on $\mathbb{C}_{N\times N}$. The following beautiful theorem about the asymptotics of finite block Toeplitz matrices was proved by Widom [27].

Theorem 1.1. (see [27, Theorem 6.1]). *If $a\in(K_{2,2}^{1/2,1/2})_{N\times N}$ and the Toeplitz operators $T(a)$ and $T(\widetilde{a})$ are invertible on H^2_N, then $T(a)T(a^{-1})-I$ is of trace class and, with appropriate branches of the logarithm,*

$$\log\det T_n(a)=(n+1)\log G(a)+\log\det{}_1 T(a)T(a^{-1})+o(1)\quad as\quad n\to\infty, \tag{1}$$

where $\det_1$ *is defined in Section* 2.1 *and*

$$G(a):=\lim_{r\to 1-0}\exp\left(\frac{1}{2\pi}\int_0^{2\pi}\log\det\widehat{a}_r(e^{i\theta})d\theta\right),\quad \widehat{a}_r(e^{i\theta}):=\sum_{n=-\infty}^{\infty}a_n r^{|n|}e^{in\theta}.$$

The proof of the above result in a more general form is contained in [5, Theorem 6.11] and [7, Theorem 10.30].

Let $\lambda_1^{(n)},\dots,\lambda_{(n+1)N}^{(n)}$ denote the eigenvalues of $T_n(a)$ repeated according to their algebraic multiplicity. Let $\operatorname{sp}A$ denote the spectrum of a bounded linear operator A and $\operatorname{tr}M$ denote the trace of a matrix M. Theorem 1.1 is equivalent to the assertion

$$\sum_i\log\lambda_i^{(n)}=\operatorname{tr}\log T_n(a)=(n+1)\log G(a)+\log\det{}_1 T(a)T(a^{-1})+o(1).$$

Widom [27] noticed that Theorem 1.1 yields even a description of the asymptotic behavior of $\operatorname{tr}f(T_n(a))$ if one replaces $f(\lambda)=\log\lambda$ by an arbitrary function f analytic in an open neighborhood of the union $\operatorname{sp}T(a)\cup\operatorname{sp}T(\widetilde{a})$ (we henceforth call such f simply analytic on $\operatorname{sp}T(a)\cup\operatorname{sp}T(\widetilde{a})$).

Theorem 1.2. (see [27, Theorem 6.2]). *If $a\in(K_{2,2}^{1/2,1/2})_{N\times N}$ and if f is analytic on $\operatorname{sp}T(a)\cup\operatorname{sp}T(\widetilde{a})$, then*

$$\operatorname{tr}f(T_n(a))=(n+1)G_f(a)+E_f(a)+o(1)\quad as\quad n\to\infty, \tag{2}$$

where

$$\begin{aligned} G_f(a) &:= \frac{1}{2\pi}\int_0^{2\pi}(\operatorname{tr}f(a))(e^{i\theta})d\theta,\\ E_f(a) &:= \frac{1}{2\pi i}\int_{\partial\Omega}f(\lambda)\frac{d}{d\lambda}\log\det{}_1 T[a-\lambda]T[(a-\lambda)^{-1}]d\lambda, \end{aligned}$$

$\det_1$ *is defined in Section* 2.1*, and Ω is any bounded open set containing the set $\operatorname{sp}T(a)\cup\operatorname{sp}T(\widetilde{a})$ on the closure of which f is analytic.*

The proof of Theorem 1.2 for continuous symbols a is also given in [7, Section 10.90] and in [6, Theorem 5.6]. In the scalar case ($N=1$) Theorems 1.1 and 1.2 go back to Gabor Szegő (see [13] and historical remarks in [5, 6, 7, 15, 24]).

1.3. **Smoothness effects.** Fisher and Hartwig [11] were probably the first to draw due attention to higher order correction terms in asymptotic formulas for Toeplitz determinants. Böttcher and Silbermann [4] obtained analogs of Theorem 1.1 for symbols belonging to Hölder-Zygmund spaces $C^\gamma_{N\times N}$, $0<\gamma<\infty$. If $\gamma>1/2$, then $C^\gamma_{N\times N}$ is properly contained in $(K^{1/2,1/2}_{2,2})_{N\times N}$, and for $a\in C^\gamma_{N\times N}$, formula (1) is then valid with $o(1)$ replaced by $O(n^{1-2\gamma})$. Nowadays this result can be proved almost immediately by using the so-called Geronimo-Case-Borodin-Okounkov formula (see [8]). The author [18] proved that if $\gamma>1/2$ and $a\in C^\gamma_{N\times N}$, then (2) holds with $o(1)$ replaced by $O(n^{1-2\gamma})$. That is, for very smooth symbols the remainders in (1) and (2) go to zero with high speed (depending on the smoothness).

On the other hand, Böttcher and Silbermann [4] (see also [5, Sections 6.15–6.20] and [7, Sections 10.34–10.38]) observed that if $0<\gamma\le 1/2$, then (1) requires a correction involving additional terms and regularized operator determinants. This is the effect of "insufficient smoothness". They also studied the same problems for Wiener algebras with power weights [4], [5, Sections 6.15–6.20], [7, Sections 10.34–10.38]. Recently the author [16] extended their higher order versions of Theorem 1.1 to Wiener algebras with general weights satisfying natural submultiplicativity, monotonicity, and regularity conditions. Corresponding higher order asymptotic trace formulas are proved in [17] (see also [7, Section 10.91]).

Very recently, it was observed in [3] that the approach of [4] with some improvements of [16] is powerful enough to deliver higher order asymptotic formulas for Toeplitz determinants with symbols in generalized Krein algebras $(K^{\alpha,\beta}_{p,q})_{N\times N}$ with $1<p,q<\infty$, $0<\alpha,\beta<1$, and $1/p+1/q=\alpha+\beta\in(0,1)$. This is another example of "insufficient smoothness" because one cannot guarantee that $T(a)T(a^{-1})-I$ is of trace class whenever $a\in(K^{\alpha,\beta}_{p,q})_{N\times N}$. Notice that generalized Krein algebras contain discontinuous functions in contrast to Hölder-Zygmund spaces and weighted Wiener algebras, which consist of continuous functions only.

1.4. **About this paper.** In this paper, we will study asymptotics of Toeplitz matrices with symbols in generalized Hölder spaces following the approach of [4]. Our results improve earlier results by Böttcher and Silbermann for $C^\gamma_{N\times N}$, $0<\gamma<1$, because the scale of generalized Hölder spaces is finer than the scale of Hölder spaces C^γ, $0<\gamma<1$ (although we will not consider generalizations of the case $\gamma\ge 1$).

The paper is organized as follows. Section 2 contains definitions of Schatten-von Neumann classes and regularized operator determinants, as well as definitions of the Bari-Stechkin class and generalized Hölder spaces $\mathcal{H}^\omega$ and their subspaces $\mathcal{H}^\omega_0$. Our main results refining Theorems 1.1 and 1.2 are stated in the end of Section 2. In Section 3, we present an abstract approach from [4] (see also [16]) to higher order asymptotic formulas for block Toeplitz matrices. To apply these results it is necessary to check that the symbol admits canonical left and right bounded Wiener-Hopf factorizations, at least one of the factors is continuous, and some products of Hankel operators belong to the Schatten-von Neumann class $\mathcal{C}_m(H^2_N)$ for $m\in\mathbb{N}$. In Section 4, we collect necessary information about Wiener-Hopf factorization in decomposing algebras of continuous functions and verify that the algebras $(\mathcal{H}^\omega)_{N\times N}$ and $(\mathcal{H}^\omega_0)_{N\times N}$ have the factorization property. In Section 5, we prove simple sufficient conditions for the membership in the Schatten-von Neumann classes of products of Hankel operators with symbols in $(\mathcal{H}^\omega)_{N\times N}$. These results

are based on the classical Jackson theorem on the best uniform approximation. In Section 6, we prove our asymptotic formulas on the basis of the results of Sections 3–5.

2. Preliminaries and the main results

2.1. **Schatten-von Neumann classes and operator determinants.** Let H be a separable Hilbert space, $\mathcal{B}(H)$ be the Banach algebra of all bounded linear operators on H, $\mathcal{C}_0(H)$ be the set of all finite-rank operators, and $\mathcal{C}_\infty(H)$ be the closed two-sided ideal of all compact operators on H. Given $A \in \mathcal{B}(H)$ define $s_n(A) := \inf\{\|A - F\|_{\mathcal{B}(H)} : F \in \mathcal{C}_0(H),\ \dim F(H) \le n\}$ for $n \in \mathbb{Z}_+$. For $1 \le p < \infty$, the collection of all operators $K \in \mathcal{B}(H)$ satisfying

$$\|K\|_{\mathcal{C}_p(H)} := \left(\sum_{n\in\mathbb{Z}_+} s_n^p(K)\right)^{1/p} < \infty$$

is denoted by $\mathcal{C}_p(H)$ and referred to as a *Schatten-von Neumann class*. Note that $\mathcal{C}_\infty(H) = \{K \in \mathcal{B}(H) : s_n(K) \to 0 \text{ as } n \to \infty\}$ and

$$\|K\|_{\mathcal{C}_\infty(H)} = \sup_{n\in\mathbb{Z}_+} s_n(K) = \|K\|_{\mathcal{B}(H)}.$$

The operators belonging to $\mathcal{C}_1(H)$ are called *trace class operators*.

Let $A \in \mathcal{B}(H)$ be an operator of the form $I + K$ with $K \in \mathcal{C}_1(H)$. If $\{\lambda_j(K)\}_{j\ge0}$ denotes the sequence of the nonzero eigenvalues of K counted up to algebraic multiplicity, then the product $\prod_{j\ge0}(1 + \lambda_j(K))$ is absolutely convergent. The *determinant* of A is defined by

$$\det A = \det(I + K) = \prod_{j\ge0}(1 + \lambda_j(K)).$$

If $K \in \mathcal{C}_m(H)$, where $m \in \mathbb{N} \setminus \{1\}$, one can still define a determinant of $I + K$, but for classes larger than $\mathcal{C}_1(H)$, the above definition requires a regularization. A simple computation (see [23, Lemma 6.1]) shows that then

$$R_m(K) := (I + K)\exp\left(\sum_{j=1}^{m-1} \frac{(-K)^j}{j}\right) - I \in \mathcal{C}_1(H).$$

Thus, it is natural to define

$$\det{}_1(I + K) := \det(I + K), \quad \det{}_m(I + K) := \det(I + R_m(K)) \text{ for } m \in \mathbb{N} \setminus \{1\}.$$

One calls $\det_m(I + K)$ the *m-regularized determinant* of $A = I + K$. For more information about Schatten-von Neumann classes and regularized operator determinants, see [12, Chap. III–IV] and also [23].

2.2. **The Bari-Stechkin class.** A real-valued function φ is said to be *almost increasing* on an interval I of $\mathbb{R}$ if there is a positive constant A such that $\varphi(x) \le A\varphi(y)$ for all $x, y \in I$ such that $x \le y$. One says that $\omega : (0, \pi] \to [0, \infty)$ belongs to the *Bari-Stechkin class* (see [1, p. 493] and [14, Chap. 2, Section 2]) if ω is almost increasing on $(0, \pi]$, $\omega(x) > 0$ for all $x \in (0, \pi]$, and

$$\lim_{x\to0+0} \omega(x) = 0, \quad \sup_{x>0} \frac{1}{\omega(x)} \int_0^x \frac{\omega(y)}{y}\, dy < \infty, \quad \sup_{x>0} \frac{x}{\omega(x)} \int_x^\pi \frac{\omega(y)}{y^2}\, dy < \infty.$$

To give an example of functions in the Bari-Stechkin class, let us define inductively the sequence of functions ℓ_k on (x_k,∞) by $\ell_1(x) := \log x$, $x_1 := 1$ and for $k \in \mathbb{N}\setminus\{1\}$, $\ell_k(x) := \log(\ell_{k-1}(x))$ and x_k such that $\ell_{k-1}(x_k) = 1$. Elementary computations show that for every $\gamma \in (0,1)$ and any finite sequence $\beta_1,\dots,\beta_m \in \mathbb{R}$ there exists a set of positive constants $b_1,\dots,b_m$ such that the function

$$\omega(x) = x^\gamma \prod_{k=1}^m \ell_k^{\beta_k}\left(\frac{b_k}{x}\right), \quad 0 < x \le \pi, \tag{3}$$

belongs to the Bari-Stechkin class. In particular, $\omega(x) = x^\gamma$, $0<\gamma<1$, is a trivial example of a function in the Bari-Stechkin class.

2.3. **Generalized Hölder spaces.** The *modulus of continuity* of a bounded function $f : \mathbb{T} \to \mathbb{C}$ is defined by

$$\omega(f,x) := \sup_{|h|\le x}\sup_{y\in\mathbb{R}} |f(e^{i(y+h)}) - f(e^{iy})|, \quad 0 \le x \le \pi.$$

Let ω belong to the Bari-Stechkin class. The generalized Hölder space $\mathcal{H}^\omega$ is defined as the set of all continuous functions $f : \mathbb{T} \to \mathbb{C}$ satisfying

$$|f|_\omega := \sup_{0<x\le\pi} \frac{\omega(f,x)}{\omega(x)} < \infty.$$

We will consider also the subspace $\mathcal{H}_0^\omega$ of functions $f \in \mathcal{H}^\omega$ such that

$$\lim_{x\to 0+0} \frac{\omega(f,x)}{\omega(x)} = 0.$$

It is well known that $\mathcal{H}^\omega$ and $\mathcal{H}_0^\omega$ are Banach algebras under the norm

$$\|f\|_{\mathcal{H}^\omega} := \|f\|_C + |f|_\omega.$$

2.4. **Higher order asymptotic formulas for determinants.** For $a \in L^\infty_{N\times N}$ and $n \in \mathbb{Z}_+$, define the operators P_n and Q_n on H^2_N by

$$P_n : \sum_{k=0}^\infty a_k t^k \mapsto \sum_{k=0}^n a_k t^k, \quad Q_n := I - P_n.$$

The operator $P_n T(a) P_n : P_n H^2_N \to P_n H^2_N$ may be identified with the finite block Toeplitz matrix $T_n(a) := (a_{j-k})_{j,k=0}^n$.

If $\mathcal{A}$ is a unital algebra, then its group of all invertible elements is denoted by $G\mathcal{A}$. For $1 \le p \le \infty$, put $\overline{H^p} := \{f \in L^p : \overline{f} \in H^p\}$. Suppose

$$v_- \in (\overline{H^\infty})_{N\times N}, \qquad v_+ \in H^\infty_{N\times N}, \tag{4}$$

$$u_- \in G(\overline{H^\infty})_{N\times N}, \qquad u_+ \in GH^\infty_{N\times N}, \tag{5}$$

and define

$$b := v_- u_+^{-1}, \quad c := u_-^{-1} v_+.$$

Theorem 2.1 (Main result 1)**.** *Let ω,ψ belong to the Bari-Stechkin class. Suppose $a \in L^\infty_{N\times N}$ can be factored as $a = u_- u_+$ with*

$$u_- \in G(\mathcal{H}^\omega \cap \overline{H^\infty})_{N\times N}, \quad u_+ \in G(\mathcal{H}^\psi \cap H^\infty)_{N\times N}, \tag{6}$$

and suppose $T(\widetilde{a})$ is invertible on H^2_N. Then the following statements hold.

(a) *The function a admits a factorization $a = v_+ v_-$, where $v_- \in G(\overline{H^\infty})_{N\times N}$ and $v_+ \in GH^\infty_{N\times N}$.*

(b) *If*

$$\sum_{k=1}^{\infty}\omega\left(\frac{1}{k}\right)\psi\left(\frac{1}{k}\right)<\infty, \tag{7}$$

then $T(a)T(a^{-1})-I$ *and* $T(\widetilde{c})T(\widetilde{b})-I$ *belong to* $\mathcal{C}_1(H^2_N)$ *and*

$$\lim_{n\to\infty}\frac{\det T_n(a)}{G(a)^{n+1}}=\det{}_1 T(a)T(a^{-1})=\frac{1}{\det_1 T(\widetilde{c})T(\widetilde{b})}.$$

(c) *If* $m\in\mathbb{N}\setminus\{1\}$ *and*

$$\sum_{k=1}^{\infty}\left[\omega\left(\frac{1}{k}\right)\psi\left(\frac{1}{k}\right)\right]^m<\infty, \tag{8}$$

then $T(\widetilde{c})T(\widetilde{b})-I\in\mathcal{C}_m(H^2_N)$ *and*

$$\lim_{n\to\infty}\frac{\det T_n(a)}{G(a)^{n+1}}\exp\left\{-\sum_{j=1}^{m-1}\frac{1}{j}\operatorname{tr}\left[\left(\sum_{k=0}^{m-1}F_{n,k}(b,c)\right)^j\right]\right\}=\frac{1}{\det_m T(\widetilde{c})T(\widetilde{b})}, \tag{9}$$

where

$$F_{n,k}(b,c):=P_nT(c)Q_n\big(Q_nH(b)H(\widetilde{c})Q_n\big)^kQ_nT(b)P_n \quad (n,k\in\mathbb{Z}_+).$$

(d) *Suppose* $m\in\mathbb{N}\setminus\{1\}$. *If* (8) *is fulfilled and*

$$\lim_{n\to\infty}\left\{\left[\omega\left(\frac{1}{n}\right)\psi\left(\frac{1}{n}\right)\right]^{m-1}\sum_{j=1}^{n}\omega\left(\frac{1}{j}\right)\psi\left(\frac{1}{j}\right)\right\}=0, \tag{10}$$

then one can remove $F_{n,m-1}(b,c)$ *in* (9), *that is,*

$$\lim_{n\to\infty}\frac{\det T_n(a)}{G(a)^{n+1}}\exp\left\{-\sum_{j=1}^{m-1}\frac{1}{j}\operatorname{tr}\left[\left(\sum_{k=0}^{m-2}F_{n,k}(b,c)\right)^j\right]\right\}=\frac{1}{\det_m T(\widetilde{c})T(\widetilde{b})}. \tag{11}$$

(e) *If* $m\in\mathbb{N}$ *and* (8) *is fulfilled, then there exists a nonzero constant* $E(a)$ *such that*

$$\begin{aligned}\log\det T_n(a)&=(n+1)\log G(a)+\log E(a)\\&\quad+\operatorname{tr}\left[\sum_{\ell=1}^{n}\sum_{j=1}^{m-1}\frac{1}{j}\left(\sum_{k=0}^{m-j-1}G_{\ell,k}(b,c)\right)^j\right]\\&\quad+O\left(\sum_{k=n+1}^{\infty}\left[\omega\left(\frac{1}{k}\right)\psi\left(\frac{1}{k}\right)\right]^m\right)\end{aligned} \tag{12}$$

as $n\to\infty$, *where*

$$G_{\ell,k}(b,c):=P_0T(c)Q_\ell\big(Q_\ell H(b)H(\widetilde{c})Q_\ell\big)^kQ_\ell T(b)P_0 \quad (\ell,k\in\mathbb{Z}_+).$$

(f) *If, under the assumptions of part* (e),

$$u_-\in G(\mathcal{H}^\omega_0\cap\overline{H^\infty})_{N\times N} \quad \text{or} \quad u_+\in G(\mathcal{H}^\psi_0\cap H^\infty)_{N\times N},$$

then (12) *holds with* $O(\dots)$ *replaced by* $o(\dots)$.

Let $\alpha, \beta \in (0,1)$ and $\omega(x) = x^\alpha$, $\psi(x) = x^\beta$. If $\alpha + \beta > 1$, then (7) holds. If $\alpha + \beta > 1/m$ for some $m \in \mathbb{N} \setminus \{1\}$, then (8) and (10) are fulfilled and we arrive at the theorem of Böttcher and Silbermann [7, Theorems 10.35(ii) and 10.37(ii)] for standard Hölder spaces. It seems that part (f) is new even for standard Hölder spaces.

2.5. **Refinements of the Szegő-Widom limit theorems.** The case of $\omega = \psi$ in Theorem 2.1 is of particular importance. In this case we will prove the following refinement of the Szegő-Widom limit theorems.

Theorem 2.2 (Main result 2)**.** *Let ω belong to the Bari-Stechkin class and let $\mathcal{H}$ be either $\mathcal{H}^\omega$ or $\mathcal{H}^\omega_0$. Suppose*

$$\sum_{k=1}^{\infty} \left[\omega\left(\frac{1}{k}\right)\right]^2 < \infty \tag{13}$$

and put

$$\delta(n, \mathcal{H}) := \begin{cases} O\left(\displaystyle\sum_{k=n+1}^{\infty} \left[\omega\left(\frac{1}{k}\right)\right]^2\right) & \text{if} \quad \mathcal{H} = \mathcal{H}^\omega, \\ o\left(\displaystyle\sum_{k=n+1}^{\infty} \left[\omega\left(\frac{1}{k}\right)\right]^2\right) & \text{if} \quad \mathcal{H} = \mathcal{H}^\omega_0. \end{cases}$$

(a) *We have $\mathcal{H}_{N\times N} \subset (K^{1/2,1/2}_{2,2})_{N\times N}$.*
(b) *If $a \in \mathcal{H}_{N\times N}$ and the Toeplitz operators $T(a)$ and $T(\widetilde{a})$ are invertible on H^2_N, then (1) holds with $o(1)$ replaced by $\delta(n, \mathcal{H})$.*
(c) *If $a \in \mathcal{H}_{N\times N}$ and f is analytic on $\operatorname{sp} T(a) \cup \operatorname{sp} T(\widetilde{a})$, then (2) holds with $o(1)$ replaced by $\delta(n, \mathcal{H})$.*

For $\mathcal{H}^\omega = C^\gamma$ with $\gamma \in (1/2, 1)$ and $O(n^{1-2\gamma})$ in place of $\delta(n, \mathcal{H})$, parts (a) and (b) are already in [4] (see also [8]) and part (c) is in [18]. Notice that the scale of generalized Hölder spaces is finer than the scale of standard Hölder spaces. For instance, for every $\gamma \in (0,1)$ there exist functions ω_1 and ω_2 of the form (3) such that

$$\bigcup_{0<\varepsilon<1-\gamma} C^{\gamma+\varepsilon} \subset \mathcal{H}^{\omega_1} \subset C^\gamma \subset \mathcal{H}^{\omega_2} \subset \bigcap_{0<\varepsilon<\gamma} C^{\gamma-\varepsilon},$$

where each of the embeddings is proper (see [14, Section II.3]). Hence, Theorems 2.1 and 2.2 refine corresponding results for standard Hölder spaces.

3. Higher order asymptotic formulas: the approach of Böttcher and Silbermann

3.1. **Asymptotic formulas involving regularized operator determinants.** The following result goes back to Böttcher and Silbermann [4] (see also [5, Sections 6.15 and 6.20] and [7, Sections 10.34 and 10.37]).

Theorem 3.1. *Suppose $a \in L^\infty_{N\times N}$ satisfies the following assumptions:*

(i) *there are two factorizations $a = u_- u_+ = v_+ v_-$, where $u_-, v_- \in G(\overline{H^\infty})_{N\times N}$ and $u_+, v_+ \in GH^\infty_{N\times N}$;*
(ii) *$u_- \in C_{N\times N}$ or $u_+ \in C_{N\times N}$.*

Then the following statements are true.

(a) *If* $H(a)H(\widetilde{a}^{-1}) \in \mathcal{C}_1(H^2_N)$, *then*

$$\lim_{n\to\infty} \frac{\det T_n(a)}{G(a)^{n+1}} = \det{}_1 T(a)T(a^{-1}).$$

(b) *If* $H(b)H(\widetilde{c})$ *and* $H(\widetilde{c})H(b)$ *belong to* $\mathcal{C}_m(H^2_N)$ *for some* $m \in \mathbb{N}$, *then* (9) *is fulfilled.*

(c) *If* $H(b)H(\widetilde{c})$ *and* $H(\widetilde{c})H(b)$ *belong to* $\mathcal{C}_m(H^2_N)$ *for some* $m \in \mathbb{N}\setminus\{1\}$ *and*

$$\lim_{n\to\infty} \mathrm{tr} F_{n,m-1}(b,c) = 0,$$

then (11) *holds.*

Proof. Part (a) follows from [7, Corollary 10.27]. Part (b) is proved in the present form in [16, Theorem 15]. Part (c) follows from part (b) and [16, Propositions 6,13, and 14]. □

Notice that hypothesis (ii) can be replaced by a weaker hypothesis (see [7, Section 10.34]), which allows us to work with two discontinuous factors u_- and u_+. This is useful in the case of generalized Krein algebras $(K^{\alpha,\beta}_{p,q})_{N\times N}$ (see [3]).

3.2. **Decomposition of the logarithm of Toeplitz determinants.** The following lemma is an important step in the proof of Theorems 2.1(e), (f) and Theorem 2.2. It was obtained in [4] (see also [5, Section 6.16] and [7, Section 10.34]).

Lemma 3.2. *Suppose* $a \in L^\infty_{N\times N}$ *satisfies hypotheses* (i) *and* (ii) *of Theorem* 3.1. *Suppose for all sufficiently large* n *(say,* $n \ge n_0$*) there exists a decomposition*

$$\mathrm{tr}\log\left\{I - \sum_{k=0}^{\infty} G_{n,k}(b,c)\right\} = -\mathrm{tr}\, M_n + s_n,$$

where $\{M_n\}_{n=n_0}^\infty$ *is a sequence of* $N\times N$ *matrices and* $\{s_n\}_{n=n_0}^\infty$ *is a sequence of complex numbers. If* $\sum_{n=n_0}^\infty |s_n| < \infty$, *then there exists a constant* $E(a) \ne 0$, *depending on* $\{M_n\}_{n=n_0}^\infty$ *and arbitrarily chosen* $N\times N$ *matrices* $M_1,\dots,M_{n_0-1}$, *such that for all* $n \ge n_0$,

$$\log\det T_n(a) = (n+1)\log G(a) + \mathrm{tr}(M_1 + \dots M_n) + \log E(a) + \sum_{k=n+1}^{\infty} s_k.$$

4. Wiener-Hopf factorization in decomposing algebras of continuous functions

4.1. **Definitions and general theorems.** Let $\mathcal{A}$ be a Banach algebra continuously embedded in C. Suppose $\mathcal{A}$ contains the set of all rational functions without poles on $\mathbb{T}$ and $\mathcal{A}$ is inverse closed in C, that is, if $a \in \mathcal{A}$ and $a(t) \ne 0$ for all $t \in \mathbb{T}$, then $a^{-1} \in \mathcal{A}$. The sets $\mathcal{A}_- := \mathcal{A} \cap \overline{H^\infty}$ and $\mathcal{A}_+ := \mathcal{A} \cap H^\infty$ are subalgebras of $\mathcal{A}$. The algebra $\mathcal{A}$ is said to be *decomposing* if every function $a \in \mathcal{A}$ can be represented in the form $a = a_- + a_+$ where $a_\pm \in \mathcal{A}_\pm$.

Let $\mathcal{A}$ be a decomposing algebra. A matrix function $a \in \mathcal{A}_{N\times N}$ is said to admit a *right* (resp. *left*) *Wiener-Hopf factorization in* $\mathcal{A}_{N\times N}$ if it can be represented in the form $a = a_- D a_+$ (resp. $a = a_+ D a_-$), where

$$a_\pm \in G(\mathcal{A}_\pm)_{N\times N}, \quad D(t) = \mathrm{diag}\{t^{\kappa_1},\dots,t^{\kappa_N}\}, \quad \kappa_i \in \mathbb{Z}, \quad \kappa_1 \le \dots \le \kappa_N.$$

The integers κ_i are usually called the *right* (resp. *left*) *partial indices* of a; they can be shown to be uniquely determined by a. If $\kappa_1 = \cdots = \kappa_N = 0$, then the respective Wiener-Hopf factorization is said to be *canonical*.

The following result was obtained by Budjanu and Gohberg [9, Theorem 4.3] and it is contained in [10, Chap. II, Corollary 5.1] and in [20, Theorem 5.7'].

Theorem 4.1. *Suppose the following two conditions hold for the algebra $\mathcal{A}$:*

(a) *the Cauchy singular integral operator*

$$(S\varphi)(t) := \frac{1}{\pi i} v.p. \int_{\mathbb{T}} \frac{\varphi(\tau)}{\tau - t} d\tau \quad (t \in \mathbb{T})$$

is bounded on $\mathcal{A}$;

(b) *for any function $a \in \mathcal{A}$, the operator $aS - SaI$ is compact on $\mathcal{A}$.*

Then every matrix function $a \in \mathcal{A}_{N\times N}$ such that $\det a(t) \neq 0$ for all $t \in \mathbb{T}$ admits a right Wiener-Hopf factorization in $\mathcal{A}_{N\times N}$.

Notice that (a) holds if and only if $\mathcal{A}$ is a decomposing algebra.

The following theorem follows from a more general result due to Shubin (see [20, Theorem 6.15]).

Theorem 4.2. *Let $\mathcal{A}$ be a decomposing algebra and let $\|\cdot\|$ be a norm in the algebra $\mathcal{A}_{N\times N}$. Suppose $a, d \in \mathcal{A}_{N\times N}$ admit canonical right and left Wiener-Hopf factorizations in the algebra $\mathcal{A}_{N\times N}$. Then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\|a - d\| < \delta$, then for every canonical right Wiener-Hopf factorization $a = a_-^{(r)} a_+^{(r)}$ and for every canonical left Wiener-Hopf factorization $a = a_+^{(l)} a_-^{(l)}$ one can choose a canonical right Wiener-Hopf factorization $d = d_-^{(r)} d_+^{(r)}$ and a canonical left Wiener-Hopf factorization $d = d_+^{(l)} d_-^{(l)}$ such that*

$$\|a_\pm^{(r)} - d_\pm^{(r)}\| < \varepsilon, \qquad \|[a_\pm^{(r)}]^{-1} - [d_\pm^{(r)}]^{-1}\| < \varepsilon,$$

$$\|a_\pm^{(l)} - d_\pm^{(l)}\| < \varepsilon, \qquad \|[a_\pm^{(l)}]^{-1} - [d_\pm^{(l)}]^{-1}\| < \varepsilon.$$

4.2. **Verification of the hypotheses of Theorem 4.1 for generalized Hölder spaces.**

Theorem 4.3. *Let ω belong to the Bari-Stechkin class and let $\mathcal{H}$ be either $\mathcal{H}^\omega$ or $\mathcal{H}_0^\omega$. Then*

(a) *$a \in \mathcal{H}$ is invertible in $\mathcal{H}$ if and only if $a(t) \neq 0$ for all $t \in \mathbb{T}$;*
(b) *S is bounded on $\mathcal{H}$;*
(c) *for $a \in \mathcal{H}$, the operator $aS - SaI$ is compact on $\mathcal{H}$.*

Proof. (a) Obviously, $G\mathcal{H} \subset GC$. Conversely, if $f \in GC$, then for all $x \in (0, \pi]$,

$$\omega(1/f, x) \le \|1/f\|_C^2\, \omega(f, x).$$

From this inequality we see that if $f \in GC \cap \mathcal{H}$, then $1/f \in \mathcal{H}$. Part (a) is proved.

(b) For $\mathcal{H}^\omega$ this result follows from the well known Zygmund estimate (see [28] and also [1, p. 492], [14, p. 10]):

$$\omega(Sf, x) \le c \int_0^x \frac{\omega(f, y)}{y}\, dy + cx \int_x^\pi \frac{\omega(f, y)}{y^2}\, dy, \quad 0 < x \le \pi, \tag{14}$$

with a positive constant c independent of $f \in \mathcal{H}^\omega$. For a self-contained proof of the boundedness of S on $\mathcal{H}^\omega$ (in a more general situation of moduli of smoothness $\omega_\alpha(f, x)$ of order $\alpha > 0$), see S. Samko and A. Yakubov [22, Theorem 2].

If $f \in \mathcal{H}_0^\omega$ and

$$F_1(f,x) := \int_0^x \frac{\omega(f,y)}{y}\,dy, \quad F_2(f,x) := x\int_x^\pi \frac{\omega(f,y)}{y^2}\,dy,$$

then, by [14, Section IV.4, Lemma 1],

$$\lim_{x\to 0+0} \frac{F_1(f,x)}{\omega(x)} = \lim_{x\to 0+0} \frac{F_2(f,x)}{\omega(x)} = 0. \tag{15}$$

From (14) and (15) it follows that $Sf \in \mathcal{H}_0^\omega$ whenever $f \in \mathcal{H}_0^\omega$. That is, S is bounded on $\mathcal{H}_0^\omega$, too. Part (b) is proved.

(c) For $a \in \mathcal{H}^\omega$, the compactness of $aS - SaI$ on $\mathcal{H}^\omega$ was proved by Tursunkulov [26] (see also a survey by N. Samko [21, Corollary 4.8]).

If $a \in \mathcal{H}_0^\omega$, then $aS - SaI$ is bounded on $\mathcal{H}_0^\omega$ by part (b) and is compact on $\mathcal{H}^\omega$ by what has just been said above. Since $\mathcal{H}_0^\omega \subset \mathcal{H}^\omega$, it is easy to see that the operator $aS - SaI$ is also compact on $\mathcal{H}_0^\omega$. □

4.3. **Wiener-Hopf factorization in generalized Hölder spaces.**

Theorem 4.4. *Let ω belong to the Bari-Stechkin class and let $\mathcal{H}$ be either $\mathcal{H}^\omega$ or $\mathcal{H}_0^\omega$. Suppose $a \in \mathcal{H}_{N\times N}$.*

(a) *If $T(a)$ is invertible on H_N^2, then a admits a canonical right Wiener-Hopf factorization in $\mathcal{H}_{N\times N}$.*
(b) *If $T(\widetilde{a})$ is invertible on H_N^2, then a admits a canonical left Wiener-Hopf factorization in $\mathcal{H}_{N\times N}$.*

Proof. We follow the proof of [18, Theorem 2.4].

(a) If $T(a)$ is invertible on H_N^2, then $\det a(t) \ne 0$ for all $t \in \mathbb{T}$ (see, e.g., [10, Chap. VII, Proposition 2.1]). Then, by [10, Chap. VII, Theorem 3.2], the matrix function a admits a canonical right generalized factorization in L_N^2, that is, $a = a_-a_+$, where $a_-^{\pm 1} \in (\overline{H^2})_{N\times N}$, $a_+^{\pm 1} \in H^2_{N\times N}$ (and, moreover, the operator $a_-Pa_-^{-1}I$ is bounded on L_N^2).

On the other hand, from Theorems 4.1 and 4.3 it follows that $a \in \mathcal{H}_{N\times N}$ admits a right Wiener-Hopf factorization $a = u_-Du_+$ in $\mathcal{H}_{N\times N}$. Then

$$u_-^{\pm 1} \in (\mathcal{H}_-)_{N\times N} \subset (\overline{H^2})_{N\times N}, \quad u_+^{\pm 1} \in (\mathcal{H}_+)_{N\times N} \subset H^2_{N\times N},$$

that is, $a = u_-Du_+$ is a right generalized factorization in L_N^2. By the uniqueness of the partial indices in a right generalized factorization in L_N^2 (see, e.g., [20, Corollary 2.1]), $D = 1$. Part (a) is proved.

(b) In view of Theorem 4.3(a), $a^{-1} \in \mathcal{H}_{N\times N}$. By [7, Proposition 7.19(b)], the invertibility of $T(\widetilde{a})$ on H_N^2 is equivalent to the invertibility of $T(a^{-1})$ on H_N^2. In view of part (a), there exist $f_\pm \in G(\mathcal{H}_\pm)_{N\times N}$ such that $a^{-1} = f_-f_+$. Put $v_\pm := f_\pm^{-1}$. Then $v_\pm \in G(\mathcal{H}_\pm)_{N\times N}$ and $a = v_+v_-$ is a canonical left Wiener-Hopf factorization in $\mathcal{H}_{N\times N}$. □

5. Some applications of approximation theory

5.1. **The best uniform approximation.**

For $n \in \mathbb{Z}_+$, let $\mathcal{P}^n$ be the set of all Laurent polynomials of the form

$$p(t) = \sum_{j=-n}^{n} \alpha_j t^j, \quad \alpha_j \in \mathbb{C}, \quad t \in \mathbb{T}.$$

By the Chebyshev theorem (see, e.g., [25, Section 2.2.1]), for $f \in C$ and $n \in \mathbb{Z}_+$, there is a Laurent polynomial $p_n(f) \in \mathcal{P}^n$ such that

$$\|f - p_n(f)\|_C = \inf_{p \in \mathcal{P}^n} \|f - p\|_C. \tag{16}$$

Each such polynomial $p_n(f)$ is called a polynomial of best uniform approximation.

By the Jackson theorem (see, e.g., [25, Section 5.1.2]), there exists a constant $A > 0$ such that for all $f \in C$ and all $n \in \mathbb{Z}_+$,

$$\inf_{p \in \mathcal{P}^n} \|f - p\|_C \le A\omega\left(f, \frac{1}{n+1}\right). \tag{17}$$

5.2. **Norms of truncations of Toeplitz and Hankel operators.** Let X be a Banach space. For definiteness, let the norm of $a = (a_{\alpha,\beta})_{\alpha,\beta=1}^N$ in $X_{N\times N}$ be given by

$$\|a\|_{X_{N\times N}} := N \max_{1\le\alpha,\beta\le N} \|a_{\alpha,\beta}\|_X.$$

We will simply write $\|a\|_\infty$, $\|a\|_C$, and $\|a\|_\omega$ instead of $\|a\|_{L^\infty_{N\times N}}$, $\|a\|_{C_{N\times N}}$, and $\|a\|_{(\mathcal{H}^\omega)_{N\times N}}$, respectively. Denote by $\|A\|$ the norm of a bounded linear operator A on H^2_N.

Put $\Delta_0 := P_0$ and $\Delta_j := P_j - P_{j-1}$ for $j \in \{0, \dots, n\}$.

Lemma 5.1. *Let $n \in \mathbb{Z}_+$. Suppose $v_\pm$ and $u_\pm$ satisfy* (4), (5), *and $u_\pm^{-1} \in C_{N\times N}$. Then there exists a positive constant A_N depending only on N such that for all $n \in \mathbb{Z}_+$ and all $j \in \{0, \dots, n\}$,*

$$\|Q_n T(b)\Delta_j\| \le A_N \|v_-\|_\infty \max_{1\le\alpha,\beta\le N} \omega\left([u_+^{-1}]_{\alpha,\beta}, \frac{1}{n-j+1}\right), \tag{18}$$

$$\|\Delta_j T(c) Q_n\| \le A_N \|v_+\|_\infty \max_{1\le\alpha,\beta\le N} \omega\left([u_-^{-1}]_{\alpha,\beta}, \frac{1}{n-j+1}\right), \tag{19}$$

$$\|Q_n H(b)\| \le A_N \|v_-\|_\infty \max_{1\le\alpha,\beta\le N} \omega\left([u_+^{-1}]_{\alpha,\beta}, \frac{1}{n+1}\right), \tag{20}$$

$$\|H(\widetilde{c}) Q_n\| \le A_N \|v_+\|_\infty \max_{1\le\alpha,\beta\le N} \omega\left([u_-^{-1}]_{\alpha,\beta}, \frac{1}{n+1}\right). \tag{21}$$

Proof. The idea of the proof is borrowed from [7, Theorem 10.35(ii)] (see also [18, Proposition 3.2]). Since $u_+^{-1}, v_+ \in H^\infty_{N\times N}$ and $u_-^{-1}, v_- \in (\overline{H^\infty})_{N\times N}$, by [7, Proposition 2.14],

$$T(b) = T(v_-)T(u_+^{-1}), \qquad T(c) = T(u_-^{-1})T(v_+),$$
$$H(b) = T(v_-)H(u_+^{-1}), \qquad H(\widetilde{c}) = H(\widetilde{u_-^{-1}})T(v_+).$$

It is easy to see that $Q_n T(v_-) P_n = 0$ and $P_n T(v_+) Q_n = 0$. Hence

$$Q_n T(b)\Delta_j = Q_n T(v_-) Q_n T(u_+^{-1})\Delta_j, \tag{22}$$

$$\Delta_j T(c) Q_n = \Delta_j T(u_-^{-1}) Q_n T(v_+) Q_n, \tag{23}$$

$$Q_n H(b) = Q_n T(v_-) Q_n H(u_+^{-1}), \tag{24}$$

$$H(\widetilde{c}) Q_n = H(\widetilde{u_-^{-1}}) Q_n T(v_+) Q_n. \tag{25}$$

Let $p_{n-j}(u_+^{-1})$ and $p_{n-j}(u_-^{-1})$ be the polynomials in $\mathcal{P}^{n-j}_{N\times N}$ of best uniform approximation of u_+^{-1} and u_-^{-1}, respectively, where $j \in \{0, \dots, n\}$. Simple computations

show that

$$Q_nT[p_{n-j}(u_+^{-1})]\Delta_j=0,\quad \Delta_jT[p_{n-j}(u_-^{-1})]Q_n=0 \tag{26}$$

for all $j\in\{0,\dots,n\}$ and

$$Q_nH[p_n(u_+^{-1})]=0,\quad H[(p_n(u_-^{-1}))\tilde{\ }]Q_n=0. \tag{27}$$

From (22) and (26) we get

$$\begin{aligned}\|Q_nT(b)\Delta_j\| &\le \|Q_nT(v_-)Q_n\|\,\|Q_nT[u_+^{-1}-p_{n-j}(u_+^{-1})]\Delta_j\|\\ &\le \text{const}\,\|v_-\|_\infty\,\|u_+^{-1}-p_{n-j}(u_+^{-1})\|_C.\end{aligned}$$

Combining this inequality with (16)–(17), we arrive at (18). Inequalities (19)–(21) can be obtained in the same way by combining (26)–(27) and representations (23)–(25), respectively. □

5.3. **The asymptotic of the trace of $F_{n,m-1}(b,c)$.**

Lemma 5.2. *Let ω,ψ belong to the Bari-Stechkin class. Suppose $v_\pm$ and $u_\pm$ satisfy* (4) *and* (6)*. If $m\in\mathbb{N}\setminus\{1\}$ and* (10) *is fulfilled, then $\mathrm{tr}F_{n,m-1}(b,c)\to 0$ as $n\to\infty$.*

Proof. Since $\Delta_jF_{n,m-1}(b,c)\Delta_j$ is an $N\times N$ matrix for each $n\in\mathbb{Z}_+$ and each $j\in\{0,\dots,n\}$, we have

$$|\mathrm{tr}\Delta_jF_{n,m-1}(b,c)\Delta_j|\le C_N\|\Delta_jF_{n,m-1}(b,c)\Delta_j\|,$$

where C_N is a positive constant depending only on N. Hence

$$|\mathrm{tr}F_{n,m-1}(b,c)|=\left|\mathrm{tr}\sum_{j=0}^{n}\Delta_jF_{n,m-1}(b,c)\Delta_j\right|\le C_N\sum_{j=0}^{n}\|\Delta_jF_{n,m-1}(b,c)\Delta_j\|. \tag{28}$$

Taking into account that $\Delta_jP_n=P_n\Delta_j=\Delta_j$ for $j\in\{0,\dots,n\}$, we obtain

$$\|\Delta_jF_{n,m-1}(b,c)\Delta_j\|\le\|\Delta_jT(c)Q_n\|\left(\|Q_nH(b)\|\,\|H(\widetilde{c})Q_n\|\right)^{m-1}\|Q_nT(b)\Delta_j\|. \tag{29}$$

From Lemma 5.1 and the definition of the semi-norms $|\cdot|_\omega$ and $|\cdot|_\psi$ it follows that for all $n\in\mathbb{Z}_+$ and all $j\in\{0,\dots,n\}$,

$$\|Q_nT(b)\Delta_j\| \le A_N\|v_-\|_\infty\left(\max_{1\le\alpha,\beta\le N}|[u_+^{-1}]_{\alpha,\beta}|_\psi\right)\psi\left(\frac{1}{n-j+1}\right), \tag{30}$$

$$\|\Delta_jT(c)Q_n\| \le A_N\|v_+\|_\infty\left(\max_{1\le\alpha,\beta\le N}|[u_-^{-1}]_{\alpha,\beta}|_\omega\right)\omega\left(\frac{1}{n-j+1}\right), \tag{31}$$

$$\|Q_nH(b)\| \le A_N\|v_-\|_\infty\left(\max_{1\le\alpha,\beta\le N}|[u_+^{-1}]_{\alpha,\beta}|_\psi\right)\psi\left(\frac{1}{n+1}\right), \tag{32}$$

$$\|H(\widetilde{c})Q_n\| \le A_N\|v_+\|_\infty\left(\max_{1\le\alpha,\beta\le N}|[u_-^{-1}]_{\alpha,\beta}|_\omega\right)\omega\left(\frac{1}{n+1}\right). \tag{33}$$

Combining (28)–(33), we get

$$|\mathrm{tr}F_{n,m-1}(b,c)|=O\left(\left[\omega\left(\frac{1}{n+1}\right)\psi\left(\frac{1}{n+1}\right)\right]^{m-1}\sum_{j=1}^{n+1}\omega\left(\frac{1}{j}\right)\psi\left(\frac{1}{j}\right)\right)$$

as $n\to\infty$. This implies that if (10) holds, then $\mathrm{tr}F_{n,m-1}(b,c)\to 0$ as $n\to\infty$. □

5.4. Products of Hankel operators in Schatten-von Neumann classes.

Lemma 5.3. *Let $1 \le p < \infty$, let ω, ψ belong to the Bari-Stechkin class, and let*

$$\sum_{k=1}^{\infty} \left[\omega\left(\frac{1}{k}\right)\psi\left(\frac{1}{k}\right)\right]^p < \infty.$$

(a) *Suppose $a \in L^\infty_{N\times N}$ admits a factorization $a = u_- u_+$ with $u_\pm$ satisfying* (6)*. Then $H(a)H(\widetilde{a}^{-1}) \in \mathcal{C}_p(H^2_N)$.*
(b) *Suppose $v_\pm$ and $u_\pm$ satisfy* (4) *and* (6)*. Then $H(\widetilde{c})H(b)$ and $H(b)H(\widetilde{c})$ belong to $\mathcal{C}_p(H^2_N)$.*

Proof. This statement is proved by analogy with [7, Lemma 10.36]. Let us prove only part (a). By [7, Proposition 2.14], $H(a) = T(u_-)H(u_+)$ and $H(\widetilde{a}^{-1}) = T(\widetilde{u_+^{-1}})H(u_-^{-1})$. For $n \in \mathbb{Z}_+$, let $p_n(u_+)$ and $p_n(u_-^{-1})$ be the polynomials in $\mathcal{P}^n_{N\times N}$ of best uniform approximation of u_+ and u_-^{-1}, respectively. Observe that

$$\dim \operatorname{Im}(T(u_-)H[p_n(u_+)]) \le n+1, \quad \dim \operatorname{Im}(T(\widetilde{u_+^{-1}})H[p_n(u_-^{-1})]) \le n+1,$$

whence

$$s_{n+1}(H(a)) \le \|T(u_-)H(u_+) - T(u_-)H[p_n(u_+)]\| \le O(\|u_+ - p_n(u_+)\|_C) \tag{34}$$

and similarly

$$s_{n+1}(H(\widetilde{a}^{-1})) \le O(\|u_-^{-1} - p_n(u_-^{-1})\|_C). \tag{35}$$

From (16), (17), and the definition of the seminorms $|\cdot|_\omega$ and $|\cdot|_\psi$ it follows that

$$\|u_+ - p_n(u_+)\|_C \le A_N \left(\max_{1\le\alpha,\beta\le N} |[u_+]_{\alpha,\beta}|_\psi\right)\psi\left(\frac{1}{n+1}\right), \tag{36}$$

$$\|u_-^{-1} - p_n(u_-^{-1})\|_C \le A_N \left(\max_{1\le\alpha,\beta\le N} |[u_-^{-1}]_{\alpha,\beta}|_\omega\right)\omega\left(\frac{1}{n+1}\right), \tag{37}$$

where A_N is a positive constant depending only on ω, ψ, N. Combining (34)–(37), we get

$$s_n(H(a)) = O\big(\psi(1/n)\big), \quad s_n(H(\widetilde{a}^{-1})) = O\big(\omega(1/n)\big) \quad (n \in \mathbb{N}). \tag{38}$$

From (38) and Horn's theorem (see, e.g. [12, Chap. II, Theorem 4.2]) it follows that

$$\sum_{k=1}^{\infty} s_k^p\big(H(a)H(\widetilde{a}^{-1})\big) \le \sum_{k=1}^{\infty} s_k^p\big(H(a)\big)s_k^p\big(H(\widetilde{a}^{-1})\big) = O\left(\sum_{k=1}^{\infty}\left[\omega\left(\frac{1}{k}\right)\psi\left(\frac{1}{k}\right)\right]^p\right),$$

which finishes the proof of part (a). Part (b) is proved similarly. □

6. Proofs of the main results

6.1. Decomposition of the trace of the logarithm of one matrix series.

Lemma 6.1. *Let ω, ψ belong to the Bari-Stechkin class and let Σ be a compact set in the complex plane. Suppose*

$$v_- : \Sigma \to (\overline{H^\infty})_{N\times N}, \qquad v_+ : \Sigma \to H^\infty_{N\times N},$$

$$u_-^{\pm 1} : \Sigma \to (\mathcal{H}^\omega \cap \overline{H^\infty})_{N\times N}, \quad u_+^{\pm 1} : \Sigma \to (\mathcal{H}^\psi \cap H^\infty)_{N\times N}$$

are continuous functions. If $m\in\mathbb{N}$, then there exist a constant $C_m\in(0,\infty)$ and a number $n_0\in\mathbb{N}$ such that for all $\lambda\in\Sigma$ and all $n\ge n_0$,

$$\operatorname{trlog}\left\{I-\sum_{k=0}^{\infty}G_{n,k}\big(b(\lambda),c(\lambda)\big)\right\}+\operatorname{tr}\left[\sum_{j=1}^{m-1}\frac{1}{j}\left(\sum_{k=0}^{m-j-1}G_{n,k}\big(b(\lambda),c(\lambda)\big)\right)^{j}\right]=s_n(\lambda)$$

and

$$\begin{aligned}|s_n(\lambda)| \quad\le\quad & C_m\big(\|v_-(\lambda)\|_\infty\|v_+(\lambda)\|_\infty\big)^m\\ &\times\left[\max_{1\le\alpha,\beta\le N}\omega\left([u_-^{-1}(\lambda)]_{\alpha,\beta},\frac{1}{n+1}\right)\right]^m\\ &\times\left[\max_{1\le\alpha,\beta\le N}\omega\left([u_+^{-1}(\lambda)]_{\alpha,\beta},\frac{1}{n+1}\right)\right]^m.\end{aligned}$$

Proof. From Lemma 5.1 it follows that

$$\begin{aligned}\big\|G_{n,k}\big(b(\lambda),c(\lambda)\big)\big\| \quad\le\quad & \big(A_N^2\|v_-(\lambda)\|_\infty\|v_+(\lambda)\|_\infty\big)^{k+1}\\ &\times\left[\max_{1\le\alpha,\beta\le N}\omega\left([u_-^{-1}(\lambda)]_{\alpha,\beta},\frac{1}{n+1}\right)\right]^{k+1}\\ &\times\left[\max_{1\le\alpha,\beta\le N}\omega\left([u_+^{-1}(\lambda)]_{\alpha,\beta},\frac{1}{n+1}\right)\right]^{k+1}.\end{aligned}$$

for all $n,k\in\mathbb{Z}_+$ and all $\lambda\in\Sigma$. Moreover,

$$\begin{aligned}&A_N^2\|v_-(\lambda)\|_\infty\|v_+(\lambda)\|_\infty\\ &\quad\times\left[\max_{1\le\alpha,\beta\le N}\omega\left([u_-^{-1}(\lambda)]_{\alpha,\beta},\frac{1}{n+1}\right)\right]\left[\max_{1\le\alpha,\beta\le N}\omega\left([u_+^{-1}(\lambda)]_{\alpha,\beta},\frac{1}{n+1}\right)\right]\\ &\le A_N^2\max_{\lambda\in\Sigma}\big(\|v_-(\lambda)\|_\infty\|v_+(\lambda)\|_\infty\|u_-^{-1}(\lambda)\|_\omega\|u_+^{-1}(\lambda)\|_\psi\big)\omega\left(\frac{1}{n+1}\right)\psi\left(\frac{1}{n+1}\right).\end{aligned}$$

Since $\omega(1/n)\to0$ and $\psi(1/n)\to0$ as $n\to\infty$, there exists a number $n_0\in\mathbb{N}$ such that the left-hand side of the latter inequality is less than one for all $\lambda\in\Sigma$ and all $n\ge n_0$. Now the proof can be developed by analogy with [17, Proposition 3.3]. □

6.2. **Proof of Theorem 2.1.**

Proof of part (a). Since ω and ψ belong to the Bari-Stechkin class, there exist $\alpha,\beta\in(0,1)$ such that $\omega(x)/x^\alpha$ and $\psi(x)/x^\beta$ are almost increasing (see [1, Lemma 2] or [14, p. 54]). Hence there exists a constant $A>0$ such that $\omega(x)\le Ax^\gamma$ and $\psi(x)\le Ax^\gamma$ for all $x\in(0,\pi]$, where $\gamma:=\min\{\alpha,\beta\}\in(0,1)$. Therefore $u_-\in(\mathcal{H}^\omega)_{N\times N}\subset C^\gamma_{N\times N}$ and $u_+\in(\mathcal{H}^\psi)_{N\times N}\subset C^\gamma_{N\times N}$, where C^γ is the standard Hölder space generated by $h(x)=x^\gamma$. Since $T(\widetilde{a})$ is invertible on H^2_N and $a=u_-u_+\in C^\gamma_{N\times N}$, by Theorem 4.4(b), the function a admits a canonical left Wiener-Hopf factorization $a=v_+v_-$ in $C^\gamma_{N\times N}$. In particular, $v_-\in G(\overline{H^\infty})_{N\times N}$ and $v_+\in GH^\infty_{N\times N}$. □

Proof of parts (b) *and* (c). From Theorem 2.1(a) it follows that hypotheses (i) and (ii) of Theorem 3.1 are satisfied. Suppose $m\in\mathbb{N}$ and (8) holds. In view of Lemma 5.3 and [7, Proposition 2.14],

$$I-T(a)T(a^{-1})=H(a)H(\widetilde{a}^{-1})\in\mathcal{C}_m(H^2_N),\quad I-T(\widetilde{c})T(\widetilde{b})=H(\widetilde{c})H(b)\in\mathcal{C}_m(H^2_N),$$

and $H(b)H(\widetilde{c})\in\mathcal{C}_m(H^2_N)$. Hence Theorems 2.1(b) and 2.1(c) follow from Theorems 3.1(a) and 3.1(b). □

Proof of part (d). By Lemma 5.2, $\operatorname{tr} F_{n,m-1}(b,c)$ as $n \to \infty$. Hence the statement follows from the arguments of the proof of part (c) and Theorem 3.1(c). □

Proof of part (e). Suppose Σ consists of one point λ only (and we will not write the dependence on it). From Lema 6.1 it follows that there exist a positive constant C_m and a number $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$,

$$\operatorname{tr}\log\left\{I - \sum_{k=0}^{\infty} G_{n,k}(b,c)\right\} = -\operatorname{tr}\left[\sum_{j=1}^{m-1}\frac{1}{j}\left(\sum_{k=0}^{m-j-1} G_{n,k}(b,c)\right)^j\right] + s_n, \tag{39}$$

where

$$|s_n| \le C_m\left[\|v_-\|_\infty\|v_+\|_\infty\|u_-^{-1}\|_\omega\|u_+^{-1}\|_\psi\,\omega\left(\frac{1}{n+1}\right)\psi\left(\frac{1}{n+1}\right)\right]^m. \tag{40}$$

From (8) and (40) we get $\sum_{n=n_0}^{\infty}|s_n| < \infty$. Applying Lemma 3.2 to the decomposition (39), we conclude that there exists a constant $E(a) \ne 0$ such that for all $n \ge n_0$,

$$\begin{aligned}\log\det T_n(a) =&(n+1)\log G(a) + \log E(a)\\ &+ \operatorname{tr}\left[\sum_{\ell=1}^{n}\sum_{j=1}^{m-1}\frac{1}{j}\left(\sum_{k=0}^{m-j-1} G_{\ell,k}(b,c)\right)^j\right] + \sum_{k=n+1}^{\infty} s_k.\end{aligned} \tag{41}$$

From (40) we get

$$\sum_{k=n+1}^{\infty} s_k = O\left(\sum_{k=n+1}^{\infty}\left[\omega\left(\frac{1}{k}\right)\psi\left(\frac{1}{k}\right)\right]^m\right) \quad (n\to\infty). \tag{42}$$

Combining (41) and (42), we arrive at (12). Part (e) is proved. □

Proof of part (f). In view of (41), it is sufficient to show that

$$\sum_{k=n+1}^{\infty} s_k = o\left(\sum_{k=n+1}^{\infty}\left[\omega\left(\frac{1}{k}\right)\psi\left(\frac{1}{k}\right)\right]^m\right) \quad (n\to\infty). \tag{43}$$

By Lemma 6.1, for all $k \ge n_0$,

$$\begin{aligned}|s_k| \le\ & C_m\big(\|v_-\|_\infty\|v_+\|_\infty\big)^m\\ &\times\left[\max_{1\le\alpha,\beta\le N}\omega\left([u_-^{-1}]_{\alpha,\beta},\frac{1}{k+1}\right)\right]^m\\ &\times\left[\max_{1\le\alpha,\beta\le N}\omega\left([u_+^{-1}]_{\alpha,\beta},\frac{1}{k+1}\right)\right]^m.\end{aligned} \tag{44}$$

If $u_- \in G(\mathcal{H}_0^\omega \cap \overline{H^\infty})_{N\times N}$, then for every $\varepsilon > 0$ there exists a number $n_1(\varepsilon) \ge n_0$ such that for all $k \ge n_1(\varepsilon)$,

$$\max_{1\le\alpha,\beta\le N}\omega\left([u_-^{-1}]_{\alpha,\beta},\frac{1}{k+1}\right) < \varepsilon\omega\left(\frac{1}{k+1}\right). \tag{45}$$

From (44) and (45) it follows that for all $n \ge n_1(\varepsilon)$,

$$\sum_{k=n+1}^{\infty}|s_k| \le \varepsilon^m C_m\big(\|v_-\|_\infty\|v_+\|_\infty\|u_+^{-1}\|_\psi\big)^m\sum_{k=n+1}^{\infty}\left[\omega\left(\frac{1}{k}\right)\psi\left(\frac{1}{k}\right)\right]^m,$$

that is, (43) holds. If $u_+ \in G(\mathcal{H}_0^\psi \cap H^\infty)_{N\times N}$, then one can show as above that (43) is fulfilled. $\square$

6.3. **Auxiliary lemma.**

Lemma 6.2. *Let ω belong to the Bari-Stechkin class. If Σ is a compact set in the complex plane and $a : \Sigma \to (\mathcal{H}_0^\omega)_{N\times N}$ is a continuous function, then*

$$\lim_{n\to\infty}\left\{\left[\omega\left(\frac{1}{n}\right)\right]^{-1}\sup_{\lambda\in\Sigma}\max_{1\le\alpha,\beta\le N}\omega\left([a(\lambda)]_{\alpha,\beta},\frac{1}{n}\right)\right\}=0.$$

Proof. Assume the contrary. Then there exist a constant $C > 0$ and a sequence $\{n_k\}_{k=1}^\infty$ such that

$$\lim_{k\to\infty}\left\{\left[\omega\left(\frac{1}{n_k}\right)\right]^{-1}\sup_{\lambda\in\Sigma}\max_{1\le\alpha,\beta\le N}\omega\left([a(\lambda)]_{\alpha,\beta},\frac{1}{n_k}\right)\right\}=C.$$

Hence there exist a number $k_0 \in \mathbb{N}$ and a sequence $\{\lambda_k\}_{k=k_0}^\infty$ such that for all $k \ge k_0$,

$$\left[\omega\left(\frac{1}{n_k}\right)\right]^{-1}\max_{1\le\alpha,\beta\le N}\omega\left([a(\lambda_k)]_{\alpha,\beta},\frac{1}{n_k}\right)\ge\frac{C}{2}>0. \tag{46}$$

Since $\{\lambda_k\}_{k=k_0}^\infty$ is bounded, there is its convergent subsequence $\{\lambda_{k_j}\}_{j=1}^\infty$. Let λ_0 be the limit of this subsequence. Clearly, $\lambda_0 \in \Sigma$ because Σ is closed. Since the function $a : \Sigma \to (\mathcal{H}_0^\omega)_{N\times N}$ is continuous at λ_0, for every $\varepsilon \in (0, C/2)$, there exists a $\Delta > 0$ such that $|\lambda-\lambda_0| < \Delta$, $\lambda \in \Sigma$ implies $\|a(\lambda)-a(\lambda_0)\|_\omega < \varepsilon$. Because $\lambda_{k_j} \to \lambda_0$ as $j \to \infty$, for that Δ there exists a number $J \in \mathbb{N}$ such that $|\lambda_{k_j} - \lambda_0| < \Delta$ for all $j \ge J$, and thus

$$\|a(\lambda_{k_j})-a(\lambda_0)\|_\omega<\varepsilon \quad \text{for all} \quad j\ge J. \tag{47}$$

On the other hand, (46) implies that

$$\left[\omega\left(\frac{1}{n_{k_j}}\right)\right]^{-1}\max_{1\le\alpha,\beta\le N}\omega\left([a(\lambda_{k_j})]_{\alpha,\beta},\frac{1}{n_{k_j}}\right)\ge\frac{C}{2}>0 \quad \text{for all} \quad j\ge J. \tag{48}$$

It is easy to see that if $f, g \in \mathcal{H}^\omega$, then for all $x \in (0, \pi]$,

$$\frac{\omega(f,x)}{\omega(x)}\le\frac{\omega(g,x)}{\omega(x)}+|f-g|_\omega.$$

Hence, for all $j \ge J$,

$$\begin{aligned}&\left[\omega\left(\frac{1}{n_{k_j}}\right)\right]^{-1}\max_{1\le\alpha,\beta\le N}\omega\left([a(\lambda_{k_j})]_{\alpha,\beta},\frac{1}{n_{k_j}}\right)\\ &\le\left[\omega\left(\frac{1}{n_{k_j}}\right)\right]^{-1}\max_{1\le\alpha,\beta\le N}\omega\left([a(\lambda_0)]_{\alpha,\beta},\frac{1}{n_{k_j}}\right)+\|a(\lambda_{k_j})-a(\lambda_0)\|_\omega.\end{aligned} \tag{49}$$

From (47)–(49) we get for all $j \ge J$,

$$\left[\omega\left(\frac{1}{n_{k_j}}\right)\right]^{-1}\max_{1\le\alpha,\beta\le N}\omega\left([a(\lambda_0)]_{\alpha,\beta},\frac{1}{n_{k_j}}\right)\ge\frac{C}{2}-\varepsilon>0.$$

It follows that there exist a pair $\alpha_0, \beta_0 \in \{1, \dots, N\}$ and a subsequence $\{m_s\}_{s\in\mathbb{N}}$ of $\{n_{k_j}\}_{j=J}^{\infty}$ such that for all $s \in \mathbb{N}$,

$$\left[\omega\left(\frac{1}{m_s}\right)\right]^{-1} \omega\left([a(\lambda_0)]_{\alpha_0,\beta_0}, \frac{1}{m_s}\right) \geq \frac{C}{2} - \varepsilon > 0.$$

This contradicts the fact that $[a(\lambda_0)]_{\alpha_0,\beta_0} \in \mathcal{H}_0^{\omega}$. □

6.4. **Proof of Theorem 2.2.**

Proof of part (a). Similarly to the proof of Lemma 5.3 one can show that if a belongs to $(\mathcal{H}^{\omega})_{N\times N}$, then

$$s_n\big(H(a)\big) = O\big(\omega(1/n)\big), \quad s_n\big(H(\widetilde{a})\big) = O\big(\omega(1/n)\big) \quad (n \in \mathbb{N}). \tag{50}$$

Combining (13) and (50), we get $H(a), H(\widetilde{a}) \in \mathcal{C}_2(H_N^2)$. It is well known that

$$(K_{2,2}^{1/2,1/2})_{N\times N} = \big\{a \in L_{N\times N}^{\infty} \ : \ H(a), H(\widetilde{a}) \in \mathcal{C}_2(H_N^2)\big\}$$

(see [6, Section 5.1], [7, Sections 10.8–10.11]), which finishes the proof. □

Proof of part (b). By Theorem 4.4, the function a admits canonical right and left Wiener-Hopf factorizations in $\mathcal{H}_{N\times N}$. From Theorem 2.1(b) we get

$$\log\det T_n(a) = (n+1)\log G(a) + \log \det_1 T(a)T(a^{-1}) + o(1) \quad (n \to \infty). \tag{51}$$

On the other hand, from Theorem 2.1(e), (f) it follows that there exists a nonzero constant $E(a)$ such that

$$\log\det T_n(a) = (n+1)\log G(a) + \log E(a) + \delta(n, \mathcal{H}) \quad (n \to \infty). \tag{52}$$

From (51) and (52) we deduce that $E(a) = \det_1 T(a)T(a^{-1})$, that is, we arrive at (1) with $o(1)$ replaced by $\delta(n, \mathcal{H})$. □

Proof of part (c). This statement is proved by analogy with [17, Theorem 1.5] and [18, Theorem 1.4], although the idea of this proof goes back to [27, Theorem 6.2]. Let Ω be any bounded open set containing the set $\operatorname{sp} T(a) \cup \operatorname{sp} T(\widetilde{a})$ on the closure of which f is analytic and let Σ be a closed neighborhood of its boundary $\partial\Omega$ such that $\Sigma \cap (\operatorname{sp} T(a) \cup \operatorname{sp} T(\widetilde{a})) = \emptyset$. Let $\lambda \in \Sigma$. Then $T(a) - \lambda I = T[a - \lambda]$ and $T(\widetilde{a}) - \lambda I = T[(a-\lambda)^{\sim}]$ are invertible on H_N^2. By Theorem 4.4, $a - \lambda$ admits canonical right and left Wiener-Hopf factorizations $a - \lambda = u_-(\lambda)u_+(\lambda) = v_+(\lambda)v_-(\lambda)$ in $\mathcal{H}_{N\times N}$. Since $a - \lambda : \Sigma \to \mathcal{H}_{N\times N}$ is a continuous function with respect to λ, in view of Theorem 4.2, these factorizations can be chosen so that the functions

$$u_-^{\pm 1}, v_-^{\pm 1} : \Sigma \to (\mathcal{H} \cap \overline{H^{\infty}})_{N\times N}, \quad u_+^{\pm 1}, v_+^{\pm 1} : \Sigma \to (\mathcal{H} \cap H^{\infty})_{N\times N}$$

are continuous. From Lemma 6.1 with $m = 1$ it follows that there exist a constant $C_1 \in (0, \infty)$ and a number $n_0 \in \mathbb{N}$ such that for all $\lambda \in \Sigma$ and all $n \geq n_0$,

$$\begin{aligned} |s_n(\lambda)| \leq & C_1 \|v_-(\lambda)\|_\infty \|v_+(\lambda)\|_\infty \left[\max_{1\le\alpha,\beta\le N} \omega\left([u_-^{-1}(\lambda)]_{\alpha,\beta}, \frac{1}{n+1}\right)\right] \\ & \times \left[\max_{1\le\alpha,\beta\le N} \omega\left([u_+^{-1}(\lambda)]_{\alpha,\beta}, \frac{1}{n+1}\right)\right], \end{aligned} \tag{53}$$

where

$$s_n(\lambda) = \operatorname{tr}\log\left\{I - \sum_{k=0}^{\infty} G_{n,k}\big(b(\lambda), c(\lambda)\big)\right\}.$$

If $a \in (\mathcal{H}^\omega)_{N\times N}$, then from (53) we get for all $\lambda \in \Sigma$ and all $n \ge n_0$,

$$|s_n(\lambda)| \le C_1 \max_{\lambda\in\Sigma} \left(\|v_-(\lambda)\|_\omega \|v_+(\lambda)\|_\omega \|u_-^{-1}(\lambda)\|_\omega \|u_+^{-1}(\lambda)\|_\omega\right) \left[\omega\left(\frac{1}{n+1}\right)\right]^2. \tag{54}$$

If $a \in (\mathcal{H}_0^\omega)_{N\times N}$, then from Lemma 6.2 it follows that for every $\varepsilon > 0$ there exists a number $n_1(\varepsilon) \ge n_0$ such that for all $\lambda \in \Sigma$ and all $n \ge n_1(\varepsilon)$,

$$\max_{1\le\alpha,\beta\le N} \omega\left([u_\pm^{-1}(\lambda)]_{\alpha,\beta}, \frac{1}{n+1}\right) < \varepsilon\omega\left(\frac{1}{n+1}\right). \tag{55}$$

Combining (53) and (55), we obtain for all $\lambda \in \Sigma$ and all $n \ge n_1(\varepsilon)$,

$$|s_n(\lambda)| \le \varepsilon^2 C_1 \max_{\lambda\in\Sigma} \left(\|v_-(\lambda)\|_\omega \|v_+(\lambda)\|_\omega\right) \left[\omega\left(\frac{1}{n+1}\right)\right]^2. \tag{56}$$

From (54) and (56) we get for all $\lambda \in \Sigma$,

$$\sum_{k=n+1}^{\infty} |s_k(\lambda)| \le \text{const} \sum_{k=n+1}^{\infty} \left[\omega\left(\frac{1}{k}\right)\right]^2 \quad \text{if} \quad a \in (\mathcal{H}^\omega)_{N\times N},\ n \ge n_0, \tag{57}$$

$$\sum_{k=n+1}^{\infty} |s_k(\lambda)| \le \varepsilon^2 \,\text{const} \sum_{k=n+1}^{\infty} \left[\omega\left(\frac{1}{k}\right)\right]^2 \quad \text{if} \quad a \in (\mathcal{H}_0^\omega)_{N\times N},\ n \ge n_1(\varepsilon). \tag{58}$$

From Lemma 3.2 and Theorem 2.1(b) it follows that for all $\lambda \in \Sigma$ and $n \ge n_0$,

$$\log\det T_n(a-\lambda) = (n+1)\log G(a-\lambda) + \log\det{}_1 T[a-\lambda]T[(a-\lambda)^{-1}] + \sum_{k=n+1}^{\infty} s_k(\lambda).$$

Multiplying this equality by $-f'(\lambda)$ and then integrating over $\partial\Omega$ by parts, we get

$$\begin{aligned} \int_{\partial\Omega} f(\lambda)\frac{d}{d\lambda}\log\det T_n(a-\lambda)d\lambda &= (n+1)\int_{\partial\Omega} f(\lambda)\frac{d}{d\lambda}\log G(a-\lambda)d\lambda \\ &\quad + 2\pi i E_f(a) - \int_{\partial\Omega} f'(\lambda)\left(\sum_{k=n+1}^{\infty} s_k(\lambda)\right)d\lambda. \end{aligned} \tag{59}$$

It was obtained in the proof of [27, Theorem 6.2] (see also [6, Theorem 5.6] and [7, Section 10.90]) that

$$\frac{1}{2\pi i}\int_{\partial\Omega} f(\lambda)\frac{d}{d\lambda}\log\det T_n(a-\lambda)d\lambda = \operatorname{tr} f(T_n(a)), \tag{60}$$

$$\frac{1}{2\pi i}\int_{\partial\Omega} f(\lambda)\frac{d}{d\lambda}\log G(a-\lambda)d\lambda = G_f(a). \tag{61}$$

From (57) and (58) it follows that

$$-\int_{\partial\Omega} f'(\lambda)\left(\sum_{k=n+1}^{\infty} s_k(\lambda)\right)d\lambda = \delta(n,\mathcal{H}) \quad (n\to\infty). \tag{62}$$

Combining (59)–(62), we arrive at (2) with $o(1)$ replaced by $\delta(n,\mathcal{H})$. □

References

[1] N. K. Bari and S. B. Stechkin, *Best approximation and differential properties of two conjugate functions*. Trudy Moskovsk. Matem. Obshch., **5** (1956), 483–522 (in Russian).

[2] A. Böttcher and S. M. Grudsky, *Spectral Properties of Banded Toeplitz Operators*. SIAM, Philadelphia, PA, 2005.

[3] A. Böttcher, A. Yu. Karlovich, and B. Silbermann, *Generalized Krein algebras and asymptotics of Toeplitz determinants*. Methods of Functional Analysis and Topology, **13** (2007), no. 3, to appear. arXiv:math.FA/0612529.

[4] A. Böttcher and B. Silbermann, *Notes on the asymptotic behavior of block Toeplitz matrices and determinants*. Math. Nachr., **98** (1980), 183–210.

[5] A. Böttcher and B. Silbermann, *Invertibility and Asymptotics of Toeplitz Matrices*. Akademie-Verlag, Berlin, 1983.

[6] A. Böttcher and B. Silbermann, *Introduction to Large Truncated Toeplitz Matrices*. Springer-Verlag, New York, 1999.

[7] A. Böttcher and B. Silbermann, *Analysis of Toeplitz Operators*. 2nd edition. Springer-Verlag, Berlin, 2006.

[8] A. Böttcher and H. Widom, *Szegő via Jacobi*. Linear Algebra Appl., **419** (2006), 656–667.

[9] M. S. Budjanu and I. C. Gohberg, *General theorems on the factorization of matrix-valued functions. II. Some tests and their consequences*. Amer. Math. Soc. Transl. (2), **102** (1973), 15–26.

[10] K. F. Clancey and I. Gohberg, *Factorization of Matrix Functions and Singular Integral Operators*. Birkhäuser Verlag, Basel, 1981.

[11] M. E. Fisher, R. E. Hartwig, *Asymptotic behavior of Toeplitz matrices and determinants*. Arch. Rational Mech. Anal., **32** (1969), 190–225.

[12] I. C. Gohberg and M. G. Krein, *Introduction to the Theory of Linear Nonselfadjoint Operators*. AMS, Providence, RI, 1969.

[13] U. Grenander and G. Szegő, *Toeplitz Forms and Their Applications*. University of California Press, Berkeley, Los Angeles, 1958.

[14] A. I. Guseinov and H. Sh. Mukhtarov, *Introduiction to the Theory of Nonlienar Singular Integral Equations*. Nauka, Moscow, 1980 (in Russian).

[15] R. Hagen, S. Roch, and B. Silbermann, *C^*-Algebras and Numerical Analysis*. Marcel Dekker, Inc., New York, 2001.

[16] A. Yu. Karlovich, *Higher order asymptotics of Toeplitz determinants with symbols in weighted Wiener algebras*. J. Math. Anal. Appl., **320** (2006), 944–963.

[17] A. Yu. Karlovich, *Asymptotics of determinants and traces of Toeplitz matrices with symbols in weighted Wiener algebras*. Z. Anal. Anwendungen, **26** (2007), 43–56.

[18] A. Yu. Karlovich, *Higher order asymptotic formulas for traces of Toeplitz matrices with symbols in Hölder-Zygmund spaces*. Proceedings of IWOTA 2005, Operator Theory: Advances and Applications, to appear. arXiv:0705.0296 [math.FA].

[19] M. G. Krein, *Certain new Banach algebras and theorems of the type of the Wiener-Lévy theorems for series and Fourier integrals*. Amer. Math. Soc. Transl. (2), **93** (1970), 177–199.

[20] G. S. Litvinchuk and I. M. Spitkovsky, *Factorization of Measurable Matrix Functions*. Birkhäuser Verlag, Basel, 1987.

[21] N. Samko, *On compactness of integral operators with a generalized weak singularity in weighted spaces of continuous functions with a given continuity modulus*. Proc. Razmadze Math. Institute, **136** (2004), 91–113.

[22] S. Samko and A. Yakubov, *A Zygmund estimate for the modulus of continuity of fractional order of the conjugate function*. Izv. Vyssh. Uchebn. Zaved. (1985), no. 12, 66–72 (in Russian).

[23] B. Simon, *Notes on infinite determinants of Hilbert space operators*. Advances in Math., **24** (1977), 244–273.

[24] B. Simon, *Orthogonal Polynomials on the Unit Circle. Part 1*. AMS, Providence, RI, 2005.

[25] A. F. Timan, *Theory of Approximation of Functions of a Real Variable*. Pergamon Press, Oxford, 1963.

[26] B. Tursunkulov, *Completely continuous operators in generalized Hölder spaces*. Dokl. Akad. Nauk Uzbek. SSR (1982), no. 12, 4–6 (in Russian).

[27] H. Widom, *Asymptotic behavior of block Toeplitz matrices and determinants. II*. Advances in Math., **21** (1976), 1–29.

[28] A. Zygmund, *O module ciagłości sumy szeregu sprzężonego z szeregiem Fouriera*. Prace Mat.-Fiz., **33** (1924), 125–132 (in Polish).

Departamento de Matemática, Instituto Superior Técnico, Av. Rovisco Pais 1, 1049–001, Lisbon, Portugal

E-mail address: akarlov@math.ist.utl.pt